Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Geometry and Physics 56 (2006) 2379–2401
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Topological symmetry of forms, $N = 1$ supersymmetry and S-duality on special manifolds

Laurent Baulieu[a,1], Alessandro Tanzini[b,*]

[a] *Laboratoire de Physique Théorique et Hautes Energies, Université de Paris VI and Paris VII, France*
[b] *S.I.S.S.A., via Beirut 2/4, 34014 Trieste, Italy*

Received 10 February 2005; accepted 17 December 2005
Available online 25 January 2006

Abstract

We study the quantization of a holomorphic 2-form coupled to a Yang–Mills field on special manifolds in various dimensions, and we show that it yields twisted supersymmetric theories. The construction determines ATQFTs (Almost Topological Quantum Field Theories), that is, theories with observables that are invariant under changes of metrics belonging to restricted classes. For Kähler manifolds in four dimensions, our topological model is related to $N = 1$ Super Yang–Mills theory. Extended supersymmetries are recovered by considering the coupling with chiral multiplets. We also analyse Calabi–Yau manifolds in six and eight dimensions, and seven-dimensional G_2 manifolds of the kind recently discussed by Hitchin. We argue that the 2-form field could play an interesting rôle in the study of the conjectured S-duality in topological strings. We finally show that in the case of real forms the partition function of our topological model in six dimensions is related to the square of that of the holomorphic Chern–Simons theory, and we discuss the uplift to seven dimensions and its relation to the topological M theory.

Keywords: Topological field theories; Supersymmetry; Topological strings

1. Introduction

The idea that Poincaré supersymmetry is a "phase" of a more fundamental symmetry is appealing. In a series of earlier works, using various examples, it was shown that Poincaré

* Corresponding author. Tel.: +39 040 3787489; fax: +39 040 3787521.
E-mail addresses: baulieu@lpthe.jussieu.fr (L. Baulieu), tanzini@sissa.it (A. Tanzini).
[1] Postal address: Laboratoire de Physique Théorique et des Hautes Energies, Unité Mixte de Recherche CNRS 7589, Université Pierre et Marie Curie, boîte postale 126.4, pl. Jussieu, F–75252 PARIS Cedex 05, France.

0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2005.12.006

supersymmetry and topological symmetry are deeply related. We have shown in [1] that the field spectrum of dimensionally reduced $N = 1$ $D = 11$ supergravity can be determined in the context of an eight-dimensional gravitational Topological Quantum Field Theory (TQFT). More precisely, the equivalence of the supergravity and topological actions was shown up to quartic fermionic terms, around a $Spin(7)$ invariant vacuum. One foresees from this result that many models, which are dimensional reductions and truncations of maximal supergravity, might possibly be related by twist to topological models [2]. The BRST operator that characterizes a topological symmetry is a scalar operator which can be defined in any given curved space, while Poincaré supersymmetry is a delicate concept in curved space. Therefore, topological symmetry could be a more fundamental concept than Poincaré supersymmetry. On the other hand, in order to perform the twist operation that relates Poincaré supersymmetry and topological symmetry, one often needs to use manifolds with special holonomy. A possible derivation of supergravity from the TQFT of a 3-form in higher dimensions was discussed in [3].

On the other hand, the relevance of the eight-dimensional topological Yang–Mills theory[2] and its coupling to a 3-form, and of its dimensional reductions in lower dimensions, was emphasized for the construction of M/F theory in [4–6]. Recently, these theories received renewed interest in the context of topological string theory [7,9–11] and its possible generalizations to M theory [12–14].

In this paper, we will discuss the quantization of (holomorphic) 2-forms coupled to a Yang–Mills field on special manifolds in various dimensions. These theories are basically ATQFTs (Almost Topological Quantum Field Theories), in the sense that they are defined in terms of a classical action and a set of observables which are invariant under changes of coordinates belonging to restricted classes, for instance, reparametrizations that respect a complex structure. This is to be compared to genuine TQFT that contain observables invariant under all possible changes of metrics. Interesting cases that we will analyse in detail are Kähler manifolds in four dimensions and special manifolds in higher (six, seven and eight) dimensions. In particular in seven dimensions we will analyse G_2 manifolds of the kind recently studied by Hitchin [15]. In the case of real forms, we will show that the partition function of our topological model in six dimensions is formally equal to the square of the partition function of the holomorphic Chern–Simons theory. This indicates that this topological BF model could play some rôle in clarifying the relationship between the black hole and the topological string partition functions pointed out in [11]. As a further remark in this direction, we observe that the uplift of our BF model to some particular seven-dimensional G_2 Hitchin manifolds is related to the topological M theory recently discussed by Dijkgraaf et al. [12]. We recall that BF models in three and four dimensions are directly related to gravitational theories [19].[3] It is then natural to investigate whether a similar relationship can be found on more general grounds for the higher dimensional models that we study in this paper, by making a suitable choice of the gauge group and of the gauge-fixing conditions [20]. These relations are the generalization to higher dimensions of the description of three-dimensional gravity with a Chern–Simons model [21].

One of the original motivations for this work was to try to understand how (twisted) $N = 1$ supersymmetric theories can be directly constructed as TQFT. As we will see in the next section, this immediately leads to the introduction in the classical action of a "charged" 2-form B, valued

[2] This theory can be untwisted and "dimensionally oxidized" in the $N = 1, d = 10$ Yang–Mills theory [4].

[3] It is interesting to observe that in four dimensions these models describe perturbations around gravitational instantons. For vanishing cosmological constant, these solutions are also described by the twisted supergravity models discussed in [2].

in the adjoint representation of a Lie algebra. In these models one can also consider the coupling to chiral multiplets. If these transform in the adjoint representation, one recovers in this way also the extended supersymmetry in a twisted form.

It was noticed that the presence of a 2-form field helps in clarifying the S-duality properties of Yang–Mills theory in four dimensions [16] and of its supersymmetric extensions in various dimensions [17]. Something similar seems to happen also in the context of the topological string. In fact in the six-dimensional case we will see that by choosing different gauge-fixing conditions on the B field one can get a theory related to the B model (more precisely, to the holomorphic Chern–Simons model) or to the A model, as a twisted maximally supersymmetric Yang–Mills model. Thus one can expect the study of this model to give interesting information about the S-duality of topological string [8–10] and on the relation between Gromov–Witten and Donaldson–Thomas invariants conjectured in [18].

The organization of the paper is as follows. In Section 1 we introduce the holomorphic BF model and discuss its relationship with $N = 1$ (twisted) supersymmetry. Notice that the quantization of this model requires the use of the Batalin–Vilkoviski formalism. In Section 2 the four-dimensional case is considered, including a detailed discussion of the coupling with a chiral multiplet. In Section 3 we discuss the six-dimensional case on a Calabi–Yau threefold and show how two different quantizations yield theories related to the B model and the A model of topological string. In Section 4 we discuss the eight-dimensional theory on a Calabi–Yau fourfold and its dimensional reduction to $CY_3 \times S^1$. The eight-dimensional model is discussed also for manifolds with $SU(4)$ structure. This theory can be regarded as a generalization of the four-dimensional self-dual Yang–Mills model [22]. In the last Section, we consider the real BF model in six and seven dimensions and its relationship with the topological M theory.

2. $N = 1$ supersymmetry and the holomorphic BF theory

The standard construction of a TQFT leads to models with $N = 2$ supersymmetry. To see this, let us consider the "prototype" case of Topological Yang–Mills theory in four and eight dimensions. The relevant BRST transformations read

$$
\begin{aligned}
&\delta A_\mu = \Psi_\mu + D_\mu c \quad \delta \Psi_\mu = D_\mu \Phi - [c, \Psi_\mu] \\
&\delta c = -\Phi - \frac{1}{2}[c, c] \quad \delta \Phi = -[c, \Phi].
\end{aligned}
\tag{1}
$$

These equations stand for the geometrical identity $(\delta + d)(A + c) + \frac{1}{2}[A + c, A + c] = F + \Psi + \Phi$ [23]. There are as many components in the topological ghosts as in the gauge fields, and to gauge fix the topological freedom, one must also introduce as many antighosts as topological ghosts. The antighosts are an anticommuting antiself-dual 2-form $\kappa_{\mu\nu}$ and an anticommuting scalar η. For each one of the antighosts, there is an associated Lagrange multiplier field, and their BRST equations are

$$
\begin{aligned}
&\delta\kappa_{\mu\nu} = b_{\mu\nu} - [c, \kappa_{\mu\nu}] \quad \delta b_{\mu\nu} = -[c, b_{\mu\nu}] \\
&\delta\bar{\Phi} = \eta - [c, \bar{\Phi}] \quad \delta\eta = [c, \eta].
\end{aligned}
\tag{2}
$$

The twist operation is a mapping from these ghost and antighost fermionic degrees of freedom on a pair of spinors, which leads one to reconstruct the spinor spectrum of $N = 2$ supersymmetry, both in four and in eight dimensions. The scalar BRST operator δ can then be identified as a Lorentz scalar combination of the $N = 2$ Poincaré supersymmetry generators. However, this "twist" operation has different geometrical interpretations in four and eight dimensions. In the former case, it is a redefinition of the Euclidean Lorentz group contained in the global $SU_L(2) \times SU_R(2) \times SU(2)$ invariance of the supersymmetric theory. In the latter case, it uses the triality of eight-dimensional space. In the previous works [4–6], a constant covariant spinor has been used, which implies that one uses *Spin*(7) invariant manifolds; one can also use a manifold with $SU(4)$ holonomy.

Using self-duality equations as gauge functions, one can build a δ-exact action that provides twisted supersymmetric theories with a δ-exact energy–momentum tensor. The cohomology of the δ symmetry determines therefore a ring of topological observables, which is a subsector of the familiar set of observables for the gauge particles. The latter is selected from the cohomology of the ordinary gauge invariance.

From (1) and (2) one concludes that in these TQFT one has twice as many fermionic degrees of freedom as bosonic ones. This makes it seemingly impossible to determine $N = 1$ models as a twist of TQFTs. We may, however, look for models with a "milder" topological symmetry, such as BF models or Chern–Simons-type models, characterized by metric independent classical actions. Such actions are not boundary terms, and thus their topological symmetry cannot be as large as that displayed in (1) and (2). This might lead us to models that are twisted $N = 1$ supersymmetric theories. In this paper, we will consider the following holomorphic BF action

$$I_{n-BF} = \int_{M_{2n}} \mathrm{Tr}(B_{n,n-2} \wedge F_{0,2}) \tag{3}$$

which is defined on any complex manifold M of complex dimension n. Some aspects of the classical action (3) were studied in [24]. It would be interesting to study its quantization and possible relation with supersymmetry. For the moment we only consider some particular models which can be obtained from (3) by choosing a particular form for the field $B_{n,n-2}$. Notice that the equations of motion for this field coming from (3) imply that $B_{n,n-2}$ is a holomorphic $(n, n-2)$ form.

One might try to define a theory that is classically invariant under the following (almost) topological symmetry, which is localized in the holomorphic sector[4]:

$$\begin{aligned} &QA_m = \Psi_m + D_m c \quad Q\Psi_m = -[c, \Psi_m] \\ &Qc = -\frac{1}{2}[c, c] \quad QA_{\bar{m}} = D_{\bar{m}} c. \end{aligned} \tag{4}$$

This "heterotic" symmetry was already used [26,27,25,28] in four dimensions and in [4,29] in higher dimensions. Here we recover it as a symmetry associated with the classical action (3). If we count the ghost degrees of freedom, we have two components for Ψ_m and one for c. Notice that Ψ_m cannot have a ghost of ghost symmetry with a ghost of ghost Φ, since $Q\Psi_m = D_m\Phi - [c, \Psi_m]$ and $Qc = \Phi - \frac{1}{2}[c, c]$ would imply that $Q^2 A_{\bar{m}} \neq 0$. Modulo gauge transformations, only one degree of freedom for the field A is left free by the symmetry in (4). Moreover, if we succeed in writing a BRST gauge-fixed action for the classical symmetry in (4), this will depend on the ghosts Ψ_m and as many antighost components as there are in Ψ_m (four components). Then the number of fermion degrees of freedom will fit with those of a single Majorana spinor and we have a chance of eventually reaching $N = 1$ supersymmetry, as we will

[4] We use the standard notation where complex indices are denoted with italic letters $m, n, \bar{m}\bar{n}$, and complex coordinates are z^m and $\bar{z}^{\bar{m}}$.

explain in detail in the next section. Notice that in these models one can also recover the coupling to a chiral multiplet in the adjoint representation and the corresponding extended supersymmetry in the twisted form.

3. Four dimensions: Kähler manifold

3.1. The classical action for a BF system on a Kähler manifold

On a Kähler manifold one can define a complex structure

$$\begin{aligned} J^{mn} &= 0, \quad J^{\bar{m}\bar{n}} = 0, \\ J^{m\bar{n}} &= i g^{m\bar{n}} \end{aligned} \tag{5}$$

which allows one to introduce complex coordinates z^m and $z^{\bar{m}}$ and $1 \leq m, \bar{m} \leq N$ in 2N dimensions by

$$J^m_n z^n = i z^m, \quad J^{\bar{m}}_{\bar{n}} z^{\bar{n}} = -i z^{\bar{m}}. \tag{6}$$

In four dimensions, the action (3) reads

$$I_{cl}(A, B) = \int_{M_4} \mathrm{Tr} B_{2,0} \wedge F_{0,2} = \int_{M_4} d^4x \sqrt{g} \mathrm{Tr} \left(\epsilon^{mn\bar{m}\bar{n}} B_{mn} F_{\bar{m}\bar{n}} \right) \tag{7}$$

where $F = dA + A \wedge A$ is the curvature of the Yang–Mills field A. The equations of motion are

$$F_{\bar{m}\bar{n}} = 0 \quad \epsilon^{mn\bar{m}\bar{n}} D_{\bar{n}} B_{mn} = 0. \tag{8}$$

Classically, A_m is undetermined, $A_{\bar{m}}$ is a pure gauge and B_{mn} is holomorphic. Notice that $B_{2,0}$ has no vector gauge invariance. It counts for one propagating degree of freedom. Altogether, there are two gauge invariant degrees of freedom that are not specified classically. Modulo gauge invariance, there is a mixed propagation between A and B. The symmetries of the action (7) are

$$\begin{aligned} & Q A_m = \Psi_m + D_m c \quad Q \Psi_m = -[c, \Psi_m] \\ & Q c = -\frac{1}{2}[c, c] \quad Q A_{\bar{m}} = D_{\bar{m}} c \\ & Q B_{mn} = -[c, B_{mn}]. \end{aligned} \tag{9}$$

In the first two lines of the above equation we can recognize the symmetry (4). The geometrical interpretation of complete charged 2-forms is a non-trivial issue. However, in the language where form degree and positive and negative ghost number are unified within a bigrading, the charged 2-form can be understood as a sort of Hodge dual to the Yang–Mills field [30]. Here we only consider the (2, 0) component of such an object, and we hope to avoid the ambiguities in defining its theory. We now explain the BRST quantization of the action (7), for the purpose of inserting it in a path integral.

3.2. Quantization of the BF system on a Kähler manifold

In order to define a quantum theory, that is, a path integral, we need to gauge fix the (almost) topological symmetry of the BF system, in a way that respects the BRST symmetry associated with this symmetry. As is well known, the antiself-duality condition in four dimensions can be expressed in complex coordinates as:

$$\begin{aligned} &F_{mn} = 0, \quad F_{\bar m \bar n} = 0 \\ &J_{m\bar n} F^{m\bar n} = 0 \end{aligned} \tag{10}$$

and one has the identity

$$\mathrm{Tr}\left(F_{\bar m \bar n} F^{\bar m \bar n} + \frac{1}{2}|J_{m\bar n} F^{m\bar n}|^2\right) = \frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu} + F_{\mu\nu}\tilde F^{\mu\nu}). \tag{11}$$

Modulo ordinary gauge invariance, we have two topological freedoms, corresponding to the two components in Ψ_m. In order to perform a suitable gauge fixing for the 2-form $B_{2,0}$ and for $A_{1,0}$, which is the part of the gauge connection absent from the classical action (7), we introduce two anticommuting antighosts κ^{mn} and κ, and two Lagrange multipliers b^{mn} and b:

$$\begin{aligned} &Q\kappa^{mn} = b^{mn} \quad Qb^{mn} = 0 \\ &Q\kappa = b \quad Qb = 0. \end{aligned} \tag{12}$$

Since $Q^2 = 0$ from the beginning, we have a first-order Batalin–Vilkoviski (BV) system. However, since the treatment of the chiral multiplet in the next section will produce a non-trivial second-order BV system, we find it convenient to introduce right now BV antifields for A and B and their ghosts, antighosts and Lagrangian multipliers. The upper left marking $*$ labels antifields.[5] Let us recall that the antifield ${}^*\phi$ of a field with ghost number g has ghost number $-g-1$ and opposite statistics. For a Q-invariant BV action S one has $Q\phi = \frac{\partial_l S}{\partial\, {}^*\phi}$ and $Q\,{}^*\phi = -\frac{\partial_l S}{\partial\phi}$. The property $Q^2 = 0$ is equivalent to the master equation

$$\frac{\partial_r S}{\partial\phi}\frac{\partial_l S}{\partial\, {}^*\phi} = 0 \tag{13}$$

where (∂_r, ∂_l) indicate respectively the derivatives from the left and from the right.

The following BV action encodes at once the classical action (7) and the definition of its BRST symmetry:

$$\begin{aligned} S = \int_{M_4} d^4x\sqrt{g}\,\mathrm{Tr}\Bigg(&\frac{1}{4}\epsilon^{mn\bar m\bar n}B_{mn}F_{\bar m\bar n} + {}^*A^m(\Psi_m + D_m c) + {}^*A^{\bar m}(D_{\bar m}c) \\ &- {}^*B^{mn}[c, B_{mn}] - {}^*\Psi^m[c, \Psi_m] - \frac{1}{2}{}^*c[c,c] + {}^*\kappa_{mn}b^{mn} + {}^*\kappa\, b\Bigg). \end{aligned} \tag{14}$$

The BV master equation (13) is satisfied, which implies gauge invariance of the classical action as well as nilpotency $Q^2 = 0$ on all the fields. It is actually important to note that the invariance of the action (14) implies that

$$Q\,{}^*B^{mn} = \epsilon^{mn\bar m\bar n}F_{\bar m\bar n} - [c, {}^*B^{mn}]. \tag{15}$$

This equation will shortly play a key role in defining the coupling to scalar fields.

5 As stressed in [30], the antifields of A appear in the ghost expansion of B and vice versa.

The topological gauge fixing corresponds to the elimination of antifields by a suitable choice of a gauge function Z. The antifields are to be replaced in the path integral by the BV formula:

$$^{*}\phi = \frac{\delta Z}{\delta \phi}. \tag{16}$$

The Q-invariant observables are formally independent of the choice of Z. In particular, their mean values are expected to be independent of small changes of the metric that one must introduce to define Z.

In order to concentrate the path integral around the antiself-duality condition (10) we choose:

$$Z = \kappa^{mn}(B_{mn} - \epsilon_{mn\bar{m}\bar{n}} F^{\bar{m}\bar{n}}) + \kappa \left(\frac{1}{2} b + i J_{m\bar{n}} F^{m\bar{n}}\right). \tag{17}$$

The BV equation (16) implies that $B_{2,0}$ is eliminated in the path integral with $B_{mn} = \epsilon_{mn\bar{m}\bar{n}} F^{\bar{m}\bar{n}}$. After Gaussian integration on b, the gauge-fixed action reads

$$S^{g.f.} = \int_{M_4} d^4x \sqrt{g} \mathrm{Tr} \left(F_{\bar{m}\bar{n}} F^{\bar{m}\bar{n}} + \frac{1}{2} |J_{m\bar{n}} F^{m\bar{n}}|^2 - 2\epsilon^{\bar{m}\bar{n}pq} \kappa_{\bar{m}\bar{n}} D_p \Psi_q + i\kappa J^{\bar{m}l} D_{\bar{m}} \Psi^l \right). \tag{18}$$

Here and in the following discussions we omit the c-dependent terms in the action. In fact these terms express the covariance of the gauge-fixing conditions (10) with respect to the gauge symmetry, and vanish when these conditions are enforced. Moreover, we leave aside the standard gauge fixing of the ordinary gauge invariance $\partial^\mu A_\mu = 0$.

The action (18) can be compared with that of $N = 1$ SYM on a Kähler manifold. It is known [31] that on a complex spin manifold the complex spinors can be identified with forms $S_\pm \otimes \mathbb{C} \sim \Omega^{0,\substack{\text{even}\\ \text{odd}}}$, so that we can identify our topological ghost Ψ_m as a left-handed Weyl spinor λ_α and the topological antighosts $(\kappa_{\bar{m}\bar{n}}, \kappa)$ as the right-handed Weyl spinor $\bar{\lambda}^{\dot{\alpha}}$. More explicitly, the holonomy group of a four-dimensional Kähler manifold is locally given by $U(2) \sim SU(2)_L \times U(1)_R \subset SU(2)_L \times SU(2)_R$, so that one can naturally identify the forms $\sigma_{\mu\ \alpha \dot{1}} dx^\mu$ and $\sigma_{\mu\ \alpha \dot{2}} dx^\mu$ as (1, 0) and (0, 1) forms respectively [26,27]. Then the twist reads[6]

$$\begin{aligned}
\Psi_m &= \lambda^\alpha \sigma_{\mu\ \alpha \dot{1}} e^\mu_m, \\
\kappa_{\bar{m}\bar{n}} &= \bar{\lambda}_{\dot{\alpha}} \bar{\sigma}_{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{2}} e^\mu_{\bar{m}} e^\nu_{\bar{n}}, \\
\kappa &= \delta_{\dot{2}}{}^{\dot{\alpha}} \bar{\lambda}_{\dot{\alpha}}.
\end{aligned} \tag{19}$$

On a hyper-Kähler manifold, the twist formula can be reinterpreted by making explicit a constant spinor dependence in (19). With this change of variables, it is immediately recognized that the action (18) is the $N = 1$, $D = 4$ Yang–Mills action

$$S_{\mathrm{SYM}} = \int_{M_4} d^4x \sqrt{g} \frac{1}{4} \mathrm{Tr} \left(F_{\mu\nu} F^{\mu\nu} + F_{\mu\nu} \tilde{F}^{\mu\nu} + \bar{\lambda} \gamma^\mu D_\mu \lambda \right). \tag{20}$$

Unlike [26], we started from a classical BF system, which, eventually, gives the $N = 1$, $D = 4$ Yang–Mills theory as a microscopic theory in a twisted form. Let us note that it should be possible to cast the topological BRST symmetry in the form of conditions on curvatures

[6] We define the Euclidean σ matrices as $\sigma_\mu = (i\tau^c, \mathbf{1})$, τ^c, $c = 1, 2, 3$, being the Pauli matrices.

yielding descent equations with asymmetric holomorphic decompositions and eventually solve the cocycle equations for Q, similarly to what has been done in [23] for the Topological Yang–Mills theory.

3.3. Coupling of the BF to a chiral multiplet

Since the $N = 2$ SYM theory is an ordinary TQFT, and since its Poincaré supersymmetric version can be obtained by coupling the $N = 1$ Yang–Mills multiplet to a chiral multiplet in the adjoint representation of the gauge group, one expects to have an expression of the $N = 1$ scalar theory as an (almost) TQFT on a Kähler manifold. As we shall see, this is slightly more complicated than the $N = 1$ Super Yang–Mills theory, since it will involve the vector gauge symmetry of a (0, 2)-charged form, and thus a second-rank BV system arises.

In order to introduce the chiral multiplet, we extend the set of classical fields of the previous section as

$$B_{2,0} \to (B_{2,0}, B_{0,2}). \tag{21}$$

However, we keep the same classical action as in (7):

$$I_{cl}(A, B_{0,2}, B_{0,2}) = \int_{M_4} \mathrm{Tr}(B_{2,0} \wedge F_{0,2}) = \int_{M_4} \mathrm{Tr}(\epsilon^{mn\bar{m}\bar{n}} B_{mn} F_{\bar{m}\bar{n}}). \tag{22}$$

Having a classical action that is independent of $B_{0,2}$ is the equivalent of having the following symmetry for $B_{0,2}$

$$\begin{aligned} QB_{\bar{m}\bar{n}} &= D_{[\bar{m}}\Psi_{\bar{n}]} - [c, B_{\bar{m}\bar{n}}] - \frac{1}{4}\epsilon_{\bar{m}\bar{n}mn}[{}^*B^{mn}, \Phi] \\ Q\Psi_{\bar{m}} &= D_{\bar{m}}\Phi - [c, \Psi_{\bar{m}}] \\ Q\Phi &= -[c, \Phi]. \end{aligned} \tag{23}$$

In fact, the unique degree of freedom carried by $B_{\bar{m}\bar{n}}$ is canceled by the two degrees of freedom of the topological ghost $\Psi_{\bar{n}}$, defined modulo the ghost of ghost symmetry generated by Φ. The presence of the antifield ${}^*B^{mn}$ in (23) is necessary in order that $Q^2 = 0$, as one can verify by using (23), and (15), together with the usual BRST variation of the ghost c, $Qc = -c^2$. We thus have a second-order BV system, since the BRST variations of the fields depend linearly on the antifield. This non-trivial property justifies, a posteriori, the classical action depending on the charged (2, 0)-form $B_{2,0}$ in an “almost topological way”, as in (7), with the ordinary gauge symmetry $QA_{\bar{m}} = D_{\bar{m}}c$ and $QB_{2,0} = -[c, B_{2,0}]$. This determines the relevant Q-transformation of the antifield of the 2-form, which is eventually necessary for obtaining a closed symmetry.

The fate of the (0, 2)-form $B_{0,2}$ is to be gauge fixed and eliminated from the action, as was the case for $B_{2,0}$, but with a different gauge function. For this purpose we choose a topological antighost that is a (0, 2)-form $\kappa^{\bar{m}\bar{n}}$, with bosonic Lagrange multiplier $b^{\bar{m}\bar{n}}$. The ghost of ghost symmetry of $\Psi_{\bar{m}}$ must be gauge fixed, and we introduce a bosonic antighost $\bar{\Phi}$ with fermionic Lagrange multiplier $\bar{\eta}$. $\kappa^{\bar{m}\bar{n}}$ and $\bar{\eta}$ will be eventually untwisted and provide half of a Majorana spinor for $N = 1$ supersymmetry.

One has in fact an ordinary pyramidal structure for a 2-form gauge field,[7] which shows that $B_{\bar{m}\bar{n}}$ truly carries zero degrees of freedom, and can be consistently gauge fixed to

[7] In this table, we indicate explicitly the ghost number of the fields by a superscript. The BRST symmetry acts on the south-west direction, as indicated by the arrows.

zero

$$
\begin{array}{ccccccccc}
 & & & & B_{\bar m\bar n} & & & & \\
 & & & \swarrow & & & & & \\
 & & \Psi^{(1)}_{\bar m} & & & & \kappa^{(-1)\bar m\bar n} & & \\
 & \swarrow & & & & \swarrow & & & \\
\Phi^{(2)} & & & & b^{(0)\bar m\bar n} & & & & \bar\Phi^{(-2)} \\
 & & & & & & & \swarrow & \\
 & & & & & & \bar\eta^{(-1)} & &
\end{array}
\tag{24}
$$

The vector ghost symmetry of the charged $(0,2)$-form with ghost $\Psi^{(1)}_{\bar m}$ plays the role of a topological symmetry. In the untwisted theory, Φ and $\bar\Phi$ will be identified as the complex scalar field for the $N=1$ chiral multiplet in four dimensions. The BV action for the fields in Table (24) and their antifields is

$$
\begin{aligned}
S_{\text{matter}} = \int_{M_4} d^4x\sqrt{g}\,\mathrm{Tr}\Big(& {}^*B^{\bar m\bar n}(D_{[\bar m}\Psi_{\bar n]} - [c, B_{\bar m\bar n}] - \frac{1}{4}\epsilon_{\bar m\bar n mn}[{}^*B^{mn}, \Phi]) \\
& + {}^*\Psi^{\bar m}(D_{\bar m}\Phi - [c, \Psi_{\bar m}]) - {}^*\Phi[c, \Phi] + {}^*\kappa_{\bar m\bar n}\, b^{\bar m\bar n} + {}^*\bar\Phi\,\bar\eta\Big).
\end{aligned}
\tag{25}
$$

For consistency, we have to add to the above action the action (14) in order to properly define the variation of ${}^*B^{mn}$ as in (15).

In order to gauge fix S_{matter}, we choose the following BV gauge function:

$$
Z' = \kappa^{\bar m\bar n}B_{\bar m\bar n} + \bar\Phi D^{\bar m}\Psi_{\bar m}. \tag{26}
$$

Using the BV equation (16) and integrating on the Lagrangian multiplier $b^{\bar m\bar n}$ one gets $B_{\bar m\bar n}=0$, and finds

$$
S^{g.f.}_{\text{matter}} = \int_{M_4} d^4x\sqrt{g}\,\mathrm{Tr}\left(\kappa^{\bar m\bar n}D_{[\bar m}\Psi_{\bar n]} + \bar\Phi D^{\bar m}D_{\bar m}\Phi + \bar\eta D^{\bar m}\Psi_{\bar m}\right). \tag{27}
$$

As for the Yang–Mills supermultiplet, we can perform a mapping of the topological ghost $\Psi_{\bar m}$ and of the topological antighosts $(\kappa^{\bar m\bar n}, \bar\eta)$ on left- and right-handed spinors $(\psi^\alpha, \bar\psi_{\dot\alpha})$ respectively. With this change of variables, we now recognize that the action (27) is the $N=1$, $D=4$ chiral multiplet action

$$
S_{\text{SYM}} = \int_{M_4} d^4x\sqrt{g}\,\mathrm{Tr}\left(\bar\Phi D^\mu D_\mu\Phi + \bar\psi\gamma^\mu D_\mu\psi\right). \tag{28}
$$

The sum of the two actions (14) and (25), when the suitable gauge-fixing conditions (17) and (26) are enforced, corresponds to the twisted $N=2$ Super Yang–Mills action, with notation that is adapted to a Kähler manifold.[8]

However, the BRST algebra we discussed so far is mapped only to an $N=1$ subsector of the $N=2$ supersymmetry. The complete $N=2$ superalgebra on a Kähler manifold has been discussed in [25,27,28]. Let us briefly display how these results can be recovered in our model.

[8] In order to recover the Yukawa couplings and the quartic term in the potential $[\bar\Phi, \Phi]^2$ typical of the $N=2$ SYM one should slightly modify the gauge-fixing fermion Z' in (26), but this does not change the results on the topological observables.

In our construction, we can interchange the role of holomorphic and antiholomorphic coordinates. We can thus consider another operator $\bar{\Delta}$

$$\begin{aligned}
&\Delta A_m = \Psi_m \quad \bar{\Delta} A_m = 0 \\
&\Delta A_{\bar{m}} = 0 \quad \bar{\Delta} A_{\bar{m}} = \Psi_{\bar{m}} \\
&\Delta \Psi_m = 0 \quad \bar{\Delta} \Psi_m = D_m \Phi \\
&\Delta \Psi_{\bar{m}} = D_{\bar{m}} \Phi \quad \bar{\Delta} \Psi_{\bar{m}} = 0 \\
&\Delta \Phi = 0 \quad \bar{\Delta} \Phi = 0
\end{aligned} \tag{29}$$

where with $\Delta, \bar{\Delta}$ we indicate the equivariant BRST operators, with the ghost field c associated to the gauge symmetry set to zero. One has:

$$\begin{aligned}
&(\Delta + \bar{\Delta}) A_m = \Psi_m \\
&(\Delta + \bar{\Delta}) A_{\bar{m}} = \Psi_{\bar{m}} \\
&(\Delta + \bar{\Delta}) \Psi_m = D_m \Phi \\
&(\Delta + \bar{\Delta}) \Psi_{\bar{m}} = D_{\bar{m}} \Phi \\
&(\Delta + \bar{\Delta}) \Phi = 0.
\end{aligned} \tag{30}$$

Thus, $\Delta^2 = \bar{\Delta}^2 = 0$, and $(\Delta + \bar{\Delta})^2 = \{\Delta, \bar{\Delta}\} = \delta_\Phi$, where δ_Φ is a gauge transformation with parameter Φ. The operator $(\Delta + \bar{\Delta})$ is the topological BRST symmetry operator (for $c = 0$), corresponding to the twisted $N = 2$ supersymmetry. The classical action which is invariant under the symmetry (30) is the (real) BF action plus a "cosmological term" $\mathrm{Tr}(B \wedge B)$, with B the complete real 2-form. This field transforms as $(\Delta + \bar{\Delta}) B = D\Psi$. In a space where one cannot consistently separate holomorphic and antiholomorphic components of forms, the only admissible operation is $(\Delta + \bar{\Delta})$, which is Lorentz invariant. Then, to close the BRST symmetry, and get $\delta^2 = (Q + \bar{Q})^2 = 0$, one must redefine the BRST transformation of the Faddeev–Popov ghost c, as follows:

$$Qc = -\frac{1}{2}[c, c] \rightarrow Qc = -\Phi - \frac{1}{2}[c, c]. \tag{31}$$

In this way one recovers the complete symmetry of the Topological Yang–Mills action (1).

4. Six dimensions: Calabi–Yau threefold

On a Calabi–Yau threefold CY_3 we can use the holomorphic closed $(3, 0)$-form $\Omega_{3,0}$ and define $B_{3,1} = \Omega_{3,0} \wedge B_{0,1}$. The classical action (3) then becomes

$$I_{cl}(A, B_{0,1}) = \int_{M_6} \Omega_{3,0} \wedge \mathrm{Tr}(B_{0,1} \wedge F). \tag{32}$$

The BRST symmetry corresponding to the action (32) is

$$\begin{aligned}
&QA_m = \Psi_m + D_m c, \\
&QA_{\bar{m}} = D_{\bar{m}} c, \\
&QB_{\bar{m}} = D_{\bar{m}} \chi - [c, B_{\bar{m}}],
\end{aligned}$$

$$Q\chi = -[c, \chi],$$
$$Qc = -\frac{1}{2}[c, c]. \tag{33}$$

The invariance of the action (32) under the transformation of the B field is guaranteed by part of the Bianchi identity

$$D_{[\bar{m}} F_{\bar{n}\bar{l}]} = 0 \tag{34}$$

and the fact that Ω is closed. Notice also that the action (32) is invariant under the *complexified* gauge group $GL(N, \mathbb{C})$. The BRST symmetry (33) follows from the following Batalin–Vilkoviski action

$$S = \int_{M_6} \sqrt{g} d^6x \mathrm{Tr} \Big(\epsilon^{\bar{m}\bar{p}\bar{q}} B_{\bar{m}} F_{\bar{p}\bar{q}} + {}^*B^{\bar{m}} (D_{\bar{m}}\chi - [c, B_{\bar{m}}]) + {}^*A^{m} (\Psi_m + D_{\bar{m}} c) + {}^*A^{\bar{m}} D_{\bar{m}} c - {}^*\Psi^{m} [c, \Psi_m] + {}^*c \left(-\frac{1}{2}[c, c] \right) \Big), \tag{35}$$

where we have normalized the (3, 0)-form Ω such that $\Omega \wedge \bar{\Omega}$ is the volume form.

Let us now proceed to the quantization of the model: this can be performed in different ways, which lead us to the study of different sets of observables. If one chooses to quantize the theory around the perturbative vacuum corresponding to holomorphic flat connections, the corresponding observables will depend on the *complex* structure Ω of the manifold, as usually happens in type B topological string theories. In fact, in this case the holomorphic BF model has a deep relationship with the holomorphic Chern–Simons theory, which can be regarded as an effective action for D5 branes in type B topological strings [32]. This relationship should be a generalization of that between BF and Chern–Simons theories in three (real) dimensions [33], and it deserves further investigation.

If instead one quantizes the theory around a non-perturbative vacuum corresponding to a stable holomorphic vector bundle, one can show that the BF model corresponds to the twisted version of a supersymmetric Yang–Mills theory! In this case the observables are dependent on the *Kähler* data of the manifold, as happens in type A topological strings. In fact a direct relationship between a twisted $U(1)$ maximally supersymmetric action and the topological vertex has been shown in [7].

The study of the holomorphic BF model in the abelian case can be then useful for clarifying the issue of S-duality in topological strings pointed out in [8–10], and the relationship between Gromov–Witten and Donaldson–Thomas invariants discussed in [18]. Let us now show the details of the two quantizations.

4.1. Perturbative quantization and the B model

In this case, the symmetries are treated as ordinary gauge symmetries and fixed with transversality conditions on the $A_{\bar{m}}$ and $B_{\bar{m}}$ fields:

$$D^{\bar{m}} A_{\bar{m}} = 0$$
$$D^{\bar{m}} B_{\bar{m}} = 0. \tag{36}$$

The BV fermion corresponding to these conditions is

$$Z = \bar{\chi} D^{\bar{m}} B_{\bar{m}} + \bar{c} D^{\bar{m}} A_{\bar{m}}. \tag{37}$$

Once the gauge-fixing conditions (36) are enforced, one has a well defined mixed propagator between the A and B fields, and can use it to evaluate the path integral in a perturbative expansion. The partition function of this model in the semiclassical limit should be related to the Ray–Singer holomorphic torsion [34], similarly to what happens for the holomorphic Chern–Simons theory analysed in [35,36]. The higher order terms in the perturbative expansion should be related to other manifold invariants. It would be interesting to study these invariants along the lines of the perturbative analysis of three-dimensional Chern–Simons theory [37].

4.2. Non-perturbative quantization and the A model

The shift symmetry on the (0, 1) part of the connection gives rise to three degrees of freedom, while the symmetry on the B field gives rise to one. These are collected into the ghost fields (Ψ_m, χ) respectively. In the non-perturbative case, the gauge-fixing conditions are chosen as follows

$$
\begin{aligned}
&F_{mn} = -\frac{4}{3}\epsilon_{mnp}B^p \\
&J^{\bar{m}n}F_{\bar{m}n} = 0
\end{aligned}
\tag{38}
$$

and amount to three conditions for the first line and one for the second. The reason for the particular choice of the coefficient in the first equation of (38) will be evident shortly. Notice that the second equation in (38) reduces the complex gauge group $GL(N,\mathbb{C})$ to the unitary group $U(N)$, and as such can be considered as a partial gauge fixing for the complex gauge symmetry of the classical action (32). This has to be completed with a further gauge fixing for the unitary group, for example the ordinary Landau gauge $\partial^\mu A_\mu = 0$. We will discuss this issue in more detail in Section 5.1. The BV fermion corresponding to (38) is given by

$$
Z = \bar{\chi}^{\bar{m}\bar{n}}\left(F_{\bar{m}\bar{n}} + \frac{4}{3}\epsilon_{mnp}B^p\right) + \bar{\eta}(2iJ^{\bar{m}n}F_{\bar{m}n} - h),
\tag{39}
$$

where $(\bar{\chi}^{\bar{m}\bar{n}}, \bar{\eta})$ are the antighosts associated with the gauge-fixing conditions (38), whose BV action is given by

$$
S_{\text{aux}} = \int_{M_6}\sqrt{g}d^6x\,\text{Tr}({}^*\bar{\chi}_{\bar{m}\bar{n}}h^{\bar{m}\bar{n}} + {}^*\bar{\eta}\,h + {}^*\bar{c}\,b).
\tag{40}
$$

Eliminating the antifields by means of (16) and implementing the gauge-fixing conditions (38) by integration on the Lagrangian multipliers, we get from (35) and (40)

$$
\begin{aligned}
S^{g.f.} = \int_{M_6} d^6x\sqrt{g}\,\text{Tr}\Bigg(&-\frac{3}{2}F^{\bar{m}\bar{n}}F_{\bar{m}\bar{n}} + |J^{m\bar{n}}F_{m\bar{n}}|^2 \\
&+\ \bar{\chi}^{\bar{m}\bar{n}}D_{[\bar{m}}\Psi_{\bar{n}]} + 2i\bar{\eta}J^{\bar{m}n}D_{\bar{m}}\Psi_n + \frac{4}{3}\epsilon_{\bar{m}\bar{n}\bar{p}}\bar{\chi}^{\bar{m}\bar{n}}D^{\bar{p}}\chi\Bigg).
\end{aligned}
\tag{41}
$$

By using the identity [38]

$$
\begin{aligned}
&-\frac{1}{4}\text{Tr}(F\wedge *F) + J\wedge\text{Tr}(F\wedge F) \\
&\quad = \text{Tr}\left(-\frac{3}{2}F^{\bar{m}\bar{n}}F_{\bar{m}\bar{n}} + |J^{\bar{m}n}F_{\bar{m}n}|^2\right)
\end{aligned}
\tag{42}
$$

we can recognize in the first line of (41) the bosonic part of the $N = 1$ $D = 6$ SYM action, modulo the topological density $J \wedge \mathrm{Tr}(F \wedge F)$, where J is the Kähler 2-form. As regards the fermionic part, we can make use of the mapping between chiral fermions and complex forms $S_\pm \otimes \mathbb{C} \sim \Omega^{\mathrm{odd}}_{\mathrm{even}}$ to map the topological ghosts (Ψ_m, χ) into the right-handed spinor $\bar\lambda$ and the topological antighosts into the left-handed spinor λ. More explicitly, we can use the covariantly constant spinor ζ of the Calabi–Yau threefold to perform the mapping

$$
\begin{aligned}
&\Psi_m \to \bar\lambda \Gamma_m \zeta\\
&\chi \to \bar\lambda \zeta\\
&\bar\chi^{\bar m \bar n} \to \zeta \Gamma^{\bar m \bar n} \lambda\\
&\bar\eta \to \epsilon_{\bar m \bar n \bar p} \zeta \Gamma^{\bar m} \Gamma^{\bar n} \Gamma^{\bar p} \lambda.
\end{aligned} \tag{43}
$$

In this way, one can recognize in (41) the twisted version of the $N = 1 D = 6$ Super Yang–Mills action. In order to reproduce the $U(1)$ twisted maximally supersymmetric action discussed in [7], one has to add to the classical action (32) the higher Chern class $F \wedge F \wedge F$ and couple this theory to an hypermultiplet, with a procedure similar to that discussed in the four-dimensional case. As in Section 3.3, one has to consider the quantization of a (0, 2)-form $B_{0,2}$. The corresponding BRST complex is the same as in table (24), but now with six-dimensional fields ($\bar m, \bar n = \bar 1, \bar 2, \bar 3$). It is straightforward to realize that the fields appearing in table (24) together with the multiplet discussed in this subsection give rise exactly to the spectrum of the twisted maximally supersymmetric Yang–Mills discussed in [7]. An alternative and more economical way would be to proceed from the dimensional reduction of the eight-dimensional model that we are going to discuss in the next section. Notice that, as discussed in Section 3.3, the coupling to the hypermultiplet does not change the classical action (32). Moreover, the higher Chern class $(F)^3$ is only a boundary term which does not affect the propagation of the A and B fields. Thus the relationship with the perturbatively quantized model of the previous subsection still holds.

5. Eight dimensions

5.1. Calabi–Yau fourfold

On a Calabi–Yau fourfold we can write the following generalization of the action (7)

$$
I_{cl}(A, B_{0,2}) = \int_{M_8} \Omega_{4,0} \wedge \mathrm{Tr}(B_{0,2} \wedge F_{0,2}). \tag{44}
$$

Here $\Omega_{4,0}$ is the holomorphic covariantly *closed* (4,0)-form. This, together with part of the Bianchi identity, ensures the invariance of the classical action (44) analogously to the CY_3 case of the previous section. Also, as in the previous section, we normalize Ω such that $\Omega \wedge \bar\Omega$ is the volume element on M_8. The action (44) displays the symmetry

$$
\begin{aligned}
&QA_M = \Psi_M + D_M c\\
&QA_{\bar M} = D_{\bar M} c,\\
&QB_{\bar M \bar N} = D_{[\bar M} \chi_{\bar N]} - [c, B_{\bar M \bar N}] - \frac{1}{4} \epsilon_{\bar M \bar N \bar P \bar Q} [{}^* B^{\bar P \bar Q}, \phi],\\
&Q\chi_{\bar N} = D_{\bar N} \phi - [c, \chi_{\bar N}],
\end{aligned}
$$

$$
\begin{aligned}
&Qc = -\frac{1}{2}[c, c], \\
&Q\phi = -[c, \phi].
\end{aligned} \tag{45}
$$

Notice that c is the complexified Faddeev–Popov ghost. The BV action corresponding to (44) is given by

$$
\begin{aligned}
S = \int_{M_8} \sqrt{g} d^8x \mathrm{Tr} \Bigg(& \epsilon^{\bar{M}\bar{N}\bar{P}\bar{Q}} B_{\bar{M}\bar{N}} F_{\bar{P}\bar{Q}} \\
& + {}^*B^{\bar{M}\bar{N}} \left(D_{[\bar{M}} \chi_{\bar{N}]} - [c, B_{\bar{M}\bar{N}}] - \frac{1}{4} \epsilon_{\bar{M}\bar{N}\bar{P}\bar{Q}} \left[{}^*B^{\bar{P}\bar{Q}}, \phi \right] \right) \\
& + {}^*A^M (\Psi_M + D_{\bar{M}} c) + {}^*A^{\bar{M}} D_{\bar{M}} c + {}^*\chi^{\bar{N}} (D_{\bar{N}} \phi - [c, \chi_{\bar{N}}]) \\
& + {}^*\bar{\chi}_{\bar{M}\bar{N}} h^{\bar{M}\bar{N}} + {}^*\bar{\chi}\, h + {}^*\bar{c}\, b + {}^*\bar{\phi}\, \bar{\eta} + {}^*c \left(-\frac{1}{2}[c, c] \right) - {}^*\phi [c, \phi] \Bigg).
\end{aligned} \tag{46}
$$

The action (44) only defines the propagation of part of the gauge field, as in the case studied in Section 2. It can be gauge fixed in by imposing six complex conditions for $B_{\bar{M}\bar{N}}$

$$
\begin{aligned}
&B^+_{\bar{M}\bar{N}} = 0, \\
&B^-_{\bar{M}\bar{N}} = F^-_{\bar{M}\bar{N}},
\end{aligned} \tag{47}
$$

and a gauge fixing for $\chi_{\bar{M}}$

$$
D^{\bar{M}} \chi_{\bar{M}} = 0. \tag{48}
$$

The projection on the self-dual or antiself-dual part $B^{\pm}_{0,2}$ of the $(0, 2)$-forms can be done by using the antiholomorphic $(0, 4)$ form. The conditions (47) can be enforced by using the BRST doublets of complex antighosts and Lagrangian multipliers $(\bar{\chi}^{\bar{M}\bar{N}}, h^{\bar{M}\bar{N}})$ and $(\bar{\phi}, \bar{\chi})$ respectively. Then, as a generalization of [4], we complete the above six complex conditions for $B_{\bar{M}\bar{N}}$ with the following complex condition:

$$
D^{\bar{M}\,\mathbf{c}} A_{\bar{M}} = 0. \tag{49}
$$

The real part of (49) is the ordinary Landau gauge condition. The imaginary part gives instead a condition analogous to the second line of (38):

$$
\begin{aligned}
&\mathrm{Im} D^{\bar{M}} A_{\bar{M}} = 0 \Rightarrow J^{M\bar{N}} F_{M\bar{N}} = 0 \\
&\mathrm{Re} D^{\bar{M}} A_{\bar{M}} = 0 \Rightarrow \partial^\mu A_\mu = 0.
\end{aligned} \tag{50}
$$

The gauge-fixing fermion corresponding to the gauge conditions (47), (49) and (50) is

$$
\begin{aligned}
Z = {}& \bar{\chi}^{\bar{M}\bar{N}+} B_{\bar{M}\bar{N}} + \bar{\chi}^{\bar{M}\bar{N}-} (B_{\bar{M}\bar{N}} - 2F_{\bar{M}\bar{N}}) + \bar{\phi} D^{\bar{M}} \chi_{\bar{M}} \\
& + \bar{\chi} \left(i J^{M\bar{N}} F_{M\bar{N}} + \frac{1}{2} h \right) + \bar{c} \left(\partial^\mu A_\mu + \frac{1}{2} b \right).
\end{aligned} \tag{51}
$$

By using the BV equation (16) and enforcing the gauge conditions (47), (49) and (50) by integration on the Lagrangian multipliers, we get the wanted action, as a twisted form of the

$D = 8$ supersymmetric Yang–Mills action. Its gauge invariant part is

$$S^{g.f.} = \int_{M_8} d^8x\sqrt{g}\mathrm{Tr}\left(2F^{\bar{M}\bar{N}-}F^-_{\bar{M}\bar{N}} + \frac{1}{2}|J^{M\bar{N}}F_{M\bar{N}}|^2 + \bar{\phi}D^{\bar{M}}D_{\bar{M}}\phi\right.$$
$$\left. + \ \bar{\chi}^{\bar{M}\bar{N}-}D_{[\bar{M}}\chi_{\bar{N}]} + \bar{\chi}^{\bar{M}\bar{N}+}D_{[\bar{M}}\Psi_{\bar{N}]} + \bar{\chi}D^M\Psi_M + \bar{\eta}D^{\bar{M}}\chi_{\bar{M}}\right). \quad (52)$$

As for the previous cases, we do not display in (52) the gauge dependent part of the action. By using the identity

$$\int_{M_8} d^8x\sqrt{g}\mathrm{Tr}\left(2F^{\bar{M}\bar{N}-}F^-_{\bar{M}\bar{N}} + \frac{1}{2}|J^{M\bar{N}}F_{M\bar{N}}|^2\right) + S_0 = \frac{1}{4}\int_{M_8} d^8x\sqrt{g}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) \quad (53)$$

where $S_0 = \int_{M_8} \Omega \wedge \mathrm{Tr}(F_{0,2} \wedge F_{0,2})$ is a surface term [4], we can recognize the first line of the action (52) as the bosonic part of the $D = 8$ SYM action. As regards the fermionic part, one can use the identification of fermions with forms $S_\pm \sim \Omega^{\mathrm{even},0}_{\mathrm{odd}}$, which for $D = 8$ reads

$$S_- \sim \Omega^{1,0} \oplus \Omega^{3,0}, \quad (54)$$
$$S_+ \sim \Omega^{0,0} \oplus \Omega^{2,0} \oplus \Omega^{4,0} \quad (55)$$

to identify the topological ghosts $(\Psi_M, \chi_{\bar{M}})$ with the right-handed projection of the Majorana spinor $\bar{\lambda}^{\dot{a}}$, $\dot{a} = 1, \ldots, 8$, and the topological antighosts $(\bar{\chi}, \bar{\chi}^{\bar{M}\bar{N}}, \bar{\eta})$ with the left-handed projection λ_a. Notice in fact that on a Calabi–Yau fourfold one can use the holomorphic 4-form Ω to identify the $(3, 0)$-form appearing on the r.h.s. of (54) with the field $\chi_{\bar{M}}$ [31]. Analogously, one can identify the scalar $\bar{\eta}$ with the $(4, 0)$-form appearing in the r.h.s. of (55). More explicitly, one can use the two left-handed covariantly constant spinors (ζ_1, ζ_2) of the Calabi–Yau fourfold to identify

$$\begin{aligned} \chi^M &= \epsilon^{MNPQ}\zeta_1\Gamma_N\Gamma_P\Gamma_Q\bar{\lambda} \\ \bar{\chi}^-_{MN} &= \zeta_1\Gamma^-_{MN}\lambda \\ \bar{\chi} &= \zeta_1\lambda \end{aligned} \quad (56)$$

and

$$\begin{aligned} \Psi_M &= \zeta_2\Gamma_M\bar{\lambda} \\ \bar{\chi}^+_{MN} &= \zeta_2\Gamma^+_{MN}\lambda \\ \bar{\eta} &= \epsilon^{MPNQ}\zeta_2\Gamma_M\Gamma_N\Gamma_P\Gamma_Q\lambda. \end{aligned} \quad (57)$$

Moduli space and *Spin*(7) **theory**: the moduli space probed by the above TQFT is a holomorphic $(0, 2)$-form $\bar{D}B_{02} = 0$ and

$$F^+_{\bar{M}\bar{N}} = 0 \quad (58)$$
$$D^{\bar{M}\mathbf{c}}A_{\bar{M}} = 0. \quad (59)$$

Notice that the classical action (44) is invariant under the group of *complex* gauge transformations $GL(N, \mathbb{C})$. The moduli space described by (58) and (59) with complex gauge group $GL(N, \mathbb{C})$ should be equivalent to that described by (58) and (50) with the unitary group $U(N)$. This last moduli space is directly related with that explored by a *Spin*(7) invariant topological action [35].

In fact, using (50) one realizes that the imaginary part of (59) together with (58) amounts to seven real conditions, which fit in the **7** part of the *Spin*(7) decomposition of the (real) 2-forms $\mathbf{28} = \mathbf{7} \oplus \mathbf{21}$. The real part of (59) is the ordinary transversality condition for the unitary gauge group. Then the theory defined by the action (44) should be equivalent to that defined by the *Spin*(7) invariant action

$$I_{\Psi - BF} = \int_{M_8} \Psi \wedge \mathrm{Tr}(B \wedge F) \tag{60}$$

where Ψ is the real *Spin*(7) invariant Cayley 4-form and B, F are real 2-forms. The mapping can be done by identifying the fundamental representation of the $SU(4)$ group with the real spinor representation of *Spin*(7).

5.2. Seven dimensions: From CY_4 to $CY_3 \times S^1$

In [4], the case of writing a BRSTQFT for a G_2 manifold was directly done by starting from the topological action

$$\int_{M_7} d^7 x c^{ijk} D_i \varphi F_{jk} \tag{61}$$

where c^{ijk} stands for the G_2-invariant tensor made of octonionic structure coefficients, and φ is a Higgs field. The BRST quantization of this topological term yields a twisted version of the dimensional reduction to seven dimensions of the $D = 8$ super Yang–Mills action on a manifold with *Spin*(7) or $SU(4)$ holonomy. Basically, the topological gauge functions are the generalizations of Bogolmony equations in seven dimensions, as shown in [4].

G_2 manifolds of the kind $\Sigma_6 \times S_1$, where Σ_6 is a Calabi–Yau threefold, are of special interest both for mathematical [15] and physical [12] applications. A topological theory for such manifolds can be obtained by considering the dimensional reduction of the model discussed in the previous section, for the Calabi–Yau fourfold, although we will shortly give the classical seven-dimensional action that one can directly quantize on such manifolds.

Starting from the ATQFT for CY_4, one can set the fourth component of the gauge connection to $A_{\bar{4}} = A_7 - iL$, where 7 is the direction along the circle S_1 and L a real scalar field. (L will shortly have a special interpretation in seven dimensions.) Moreover, the dimensional reduction is imposed to set $i(\partial_4 - \partial_{\bar{4}})A_{\bar{M}} = \partial_8 A_{\bar{M}} = 0$ for any $\bar{M} = \bar{1}, \ldots, \bar{4}$. Then one gets from the classical action (44) the following action

$$S = \int_{\Sigma_6 \times S^1} \sqrt{g} d^7 x \mathrm{Tr}\left[2\epsilon^{\bar{m}\bar{n}\bar{p}}(B_{\bar{m}\bar{n}} F_{\bar{p}\bar{4}} + B_{\bar{m}} F_{\bar{n}\bar{p}})\right] \tag{62}$$

with $F_{\bar{p}\bar{4}} = F_{\bar{p}7} - iD_{\bar{p}}L$ and $B_{\bar{m}} = B_{\bar{m}\bar{4}}$. In this section we define the complex 3-form $\Omega_{0,3}$ on Σ_6 starting from the (normalized) complex 4-form in eight dimensions as $\epsilon_{\bar{m}\bar{n}\bar{p}} = \epsilon_{\bar{m}\bar{n}\bar{p}\bar{4}}$. We thus consider the following classical action:

$$I_{cl}(A, B_{0,2}) = \int_{M_7} \Omega_{3,0} \wedge \mathrm{Tr}(B \wedge F). \tag{63}$$

Here the coordinates on the manifold are $z^m, z^{\bar{m}}$ for Σ_6 and the periodic real coordinate x^7 for the circle. The only components of the 2-form that have a relevant propagation are $B_{\bar{m}\bar{n}}, B_{\bar{m}7}$.

The covariant quantization of $B_{\bar{m}\bar{n}}$, $B_{\bar{m}}(=B_{\bar{m}7})$ and A requires seven topological antighosts $(\kappa^{\bar{m}\bar{n}}, \kappa^{\bar{m}}, \kappa)$. Modulo the ordinary gauge symmetry, we have indeed seven freedoms for the classical action (62) associated with the topological ghosts $(\chi_{\bar{m}}, \chi)$ for the fields $(B_{\bar{m}\bar{n}}, B_{\bar{m}})$ and the ghost Ψ_m for the field A_m. The relevant invariance is $B_{\bar{m}\bar{n}} \sim B_{\bar{m}\bar{n}} + D_{[\bar{m}}\chi_{n]}$, $B_{\bar{m}} \sim B_{\bar{m}} + D_m\chi\, A_m \sim A_m + \Psi_m$, $A_{\bar{m}} \sim A_{\bar{m}}$, $A_7 \sim A_7$, modulo ordinary gauge transformations. Moreover, in the BRST complex for the field B there will appear also three commuting scalar ghost of ghosts $(\Phi, L, \bar{\Phi})$, for the "gauge symmetries" of the topological ghosts and antighosts of B_2. All these fields are conveniently displayed as elements of the following pyramidal diagram:

$$\begin{array}{ccccccccc}
 & & & & B_{\bar{m}\bar{n}}, B_{\bar{m}7}, A_m & & & & \\
 & & & \swarrow & & & & & \\
 & & \chi^{(1)}_{\bar{m}}, \chi^{(1)}, \Psi_m & & & & \kappa^{(-1)\bar{m}\bar{n}}, \kappa^{(-1)\bar{m}} & & \\
 & \swarrow & & & & \swarrow & & & \\
\Phi^{(2)} & & & & L^{(0)}, b^{(0)\bar{m}\bar{n}}, b^{(0)\bar{m}} & & & & \bar{\Phi}^{(-2)} \\
 & & & \swarrow & & & & \swarrow & \\
 & & \eta^{(1)} & & & & \bar{\eta}^{(-1)} & &
\end{array}$$

From the point of view of the dimensional reduction of the CY_4 theory, it is interesting to observe that the medium ghost of ghost L of the B field can be identified with the component A_8 of the eight-dimensional gauge field. The BRST transformations of the fields can be read from the following Batalin–Vilkoviski action:

$$\begin{aligned}
S = \int_{\Sigma_6\times S^1} \sqrt{g} d^7x \mathrm{Tr}\Bigg[& 2\epsilon^{\bar{m}\bar{n}\bar{p}}(B_{\bar{m}\bar{n}}F_{\bar{p}\bar{4}} + B_{\bar{m}}F_{\bar{n}\bar{p}}) \\
& + {}^*B^{\bar{m}\bar{n}}\left(D_{[\bar{m}}\chi_{\bar{n}]} - [c, B_{\bar{m}\bar{n}}] + \frac{1}{2}\epsilon_{\bar{m}\bar{n}\bar{p}}[{}^*B^{\bar{p}}, \Phi]\right) \\
& + {}^*B^{\bar{m}}\left(2D_{[\bar{m}}\chi_{7]} - 2[c, B_{\bar{m}}] + \frac{1}{2}\epsilon_{\bar{m}\bar{n}\bar{p}}[{}^*B^{\bar{n}\bar{p}}, \Phi]\right) \\
& + {}^*\chi^{\bar{m}}(D_{\bar{m}}\Phi - [c, \chi_{\bar{m}}]) + {}^*\chi(D_{\bar{4}}\Phi - [c, \chi]) \\
& + {}^*A^m(\Psi_m + D_{\bar{m}}c) + {}^*A^{\bar{m}}\, D_{\bar{m}}c + {}^*A^7\, D_7c - {}^*\Psi^m[c, \Psi_m] \\
& + {}^*c\left(-\frac{1}{2}[c, c]\right) + {}^*\bar{\Phi}\,\bar{\eta} + {}^*L\,\eta + {}^*\kappa_{\bar{m}\bar{n}}\, b^{\bar{m}\bar{n}} + 2\,{}^*\kappa_{\bar{m}}\, b^{\bar{m}} + {}^*\kappa\, b\Bigg]. \qquad (64)
\end{aligned}$$

This action is actually the dimensional reduction of action (46). However, the interpretation of the ghost of ghost system is quite different. Notice that the vectorial part of the BRST transformation for the field $B_{\bar{m}}$ is

$$\delta B_{\bar{m}} = D_{\bar{m}}\chi - D_{\bar{4}}\chi_{\bar{m}} = D_{\bar{m}}\chi - D_7\chi_{\bar{m}}. \qquad (65)$$

The variation of the action due to the first term in (65) simply vanishes after integration by parts because of the identity (34). In the context of the dimensional reduction, this identity can be read as the $\bar{M} = \bar{4}$ component of the eight-dimensional one $\epsilon^{\bar{M}\bar{N}\bar{P}\bar{Q}}D_{\bar{M}}F_{\bar{N}\bar{Q}} = 0$. The variation associated with the last term in (65), together with that coming from the variation of the field $B_{\bar{m}\bar{n}}$, $QB_{\bar{m}\bar{n}} = D_{[\bar{m}}\chi_{\bar{n}]}$, gives after integration by parts the other three components $\bar{M} = \bar{1}, \bar{2}, \bar{3}$ of the above eight-dimensional identity. The topological freedom of the classical action (62) can be fixed by choosing seven independent gauge functions. The first six can be directly obtained from the dimensional reduction of (47)

$$
\begin{aligned}
B_{\bar{m}\bar{n}} &= \frac{1}{2}(F_{\bar{m}\bar{n}} - \epsilon_{\bar{m}\bar{n}\bar{p}} F^{\bar{p}\bar{4}}) \\
B_{\bar{m}} &= \frac{1}{2}\left(F_{\bar{n}\bar{4}} - \frac{1}{2}\epsilon_{\bar{n}\bar{p}\bar{q}} F^{\bar{p}\bar{q}}\right) \\
&= \frac{1}{2}\left(F_{\bar{n}7} - i D_{\bar{n}} L - \frac{1}{2}\epsilon_{\bar{n}\bar{p}\bar{q}} F^{\bar{p}\bar{q}}\right)
\end{aligned} \tag{66}
$$

while the seventh one corresponds to the imaginary part of the complex gauge-fixing condition (49)

$$
\frac{1}{2} J^{m\bar{n}} F_{m\bar{n}} + D_7 L = 0. \tag{67}
$$

Notice in fact that the reduction on a manifold of the kind $\Sigma_6 \times S^1$ breaks the complex group of gauge invariance of the eight-dimensional action (44) to the unitary group. The residual gauge invariance under this group can be fixed by the ordinary transversality condition $\partial^{\bar{m}} A_{\bar{m}} + \partial^m A_m + \partial^7 A_7 = 0$. The gauge-fixing fermion corresponding to the conditions (66) and (67) is

$$
\begin{aligned}
Z = {} & \kappa^{\bar{m}\bar{n}}\left[B_{\bar{m}\bar{n}} - \frac{1}{2}\left(F_{\bar{m}\bar{n}} - \epsilon_{\bar{m}\bar{n}\bar{p}}(F^{\bar{p}7} + i D^{\bar{p}} L)\right)\right] \\
& + \kappa^{\bar{m}}\left(B_{\bar{m}} - \frac{3}{2}\left(F_{\bar{n}7} - i D_{\bar{n}} L - \frac{1}{2}\epsilon_{\bar{n}\bar{p}\bar{q}} F^{\bar{p}\bar{q}}\right)\right) \\
& + \kappa\left(2(J^{\bar{m}n} F_{\bar{m}n} + D_7 L) - b\right) + \bar{\Phi}(D^{\bar{m}} \chi_{\bar{m}} + D^7 \chi).
\end{aligned} \tag{68}
$$

The bosonic part of the gauge-fixed action, which can be obtained as usual by using the BV equation (16) and enforcing the gauge conditions (66) and (67) by integration on the Lagrangian multipliers, reads

$$
\begin{aligned}
S^{g.f.} = \int_{M_7} & d^7 x \sqrt{g} \mathrm{Tr}\left(-\frac{3}{2} F^{\bar{m}\bar{n}} F_{\bar{m}\bar{n}} + |J^{m\bar{n}} F_{m\bar{n}}|^2 - \frac{1}{2} F^{\bar{m}7} F_{\bar{m}7}\right. \\
& \left. + \ \bar{\Phi}(D^{\bar{m}} D_{\bar{m}} + D^7 D^7)\Phi + L(D^{\bar{m}} D_{\bar{m}} + D^7 D^7) L\right)
\end{aligned} \tag{69}
$$

and can be identified with the bosonic part of the seven-dimensional Super Yang–Mills case on $M_7 = \Sigma_6 \times S^1$.

As regards the fermionic sector, we have eight topological ghosts, $\Psi_{\bar{m}}$, $\chi_{\bar{m}}$, χ and η, and eight topological antighosts $\kappa^{\bar{m}\bar{n}}$, $\kappa^{\bar{m}}$, κ and $\bar{\eta}$. The mapping with spinors can be obtained from the dimensional reduction of the eight-dimensional mapping (56) and (57). After the dimensional reduction to $\Sigma_6 \times S^1$, the two covariantly constant eight-dimensional chiral spinors (ζ_1, ζ_2) are identified with the unique covariantly constant Majorana spinor ξ of the Calabi–Yau threefold Σ_6. On the other hand, the eight-dimensional spinor λ yields two seven-dimensional Majorana spinors (λ_1, λ_2). This results in the mapping

$$
\begin{aligned}
& \Psi_m \to \xi \Gamma_m \lambda_1, \quad \eta \to \xi \Gamma_7 \lambda_1 \\
& \chi^{\bar{m}} \to \epsilon^{mnp} \xi \Gamma_{np} \lambda_1, \quad \chi \to \epsilon^{mnp} \xi \Gamma_{mnp} \lambda_1
\end{aligned} \tag{70}
$$

for the topological ghosts and

$$\kappa_{mn} \to \xi \Gamma_{mn}\lambda_2, \quad \kappa_m \to \xi \Gamma_{m7}\lambda_2$$
$$\kappa \to \xi\lambda_2, \quad \bar{\eta} \to \epsilon^{mnp}\Gamma_{mnp7}\lambda_2 \tag{71}$$

for the topological antighosts. By this mapping we can identify the topological action (62) gauge fixed with the conditions (66) and (67) as the twisted version of the $N = 2$ seven-dimensional Super Yang–Mills case.[9] The observables of the topological model can be identified with the dimensional reduction of the eight-dimensional cocycles.

5.3. Manifolds with $SU(4)$ group structure and self-dual Yang–Mills in eight dimensions

On a Kähler manifold with a $SU(4)$ group structure we can choose $B_{4,2} = J \wedge J \wedge B^+_{2,0}$, where J is as usual the Kähler (1, 1) form. Here the 2-form $B^+_{2,0}$ is self-dual in the indices $[mn]$. Self-duality is defined from an $SU(4)$ invariant 4-form $\Omega_{4,0}$, which is globally well defined, but not necessarily closed. It counts for three degrees of freedom, according to the $SU(4)$ independent decomposition of a 2-form in eight dimensions:

$$28 = 6 \oplus \bar{6} \oplus 15 \oplus 1 \tag{72}$$

and a further decomposition 6 as $6 = 3 \oplus 3$, using the ϵ_{mnpq} tensor.

The novelty of this case is that neither the Kähler (1, 1)-form nor $\Omega_{4,0}$ are necessarily closed. Both the forms can also be rewritten in terms of spinors, which correspondingly are not covariantly constant with respect to the usual spin connection, but only with respect to a modified connection including torsion terms. The corresponding classical action is a generalization of that in (7):

$$I_{k-BF}(A, B^+_{2,0}, B^+_{0,2}) = \int_{M_8} J \wedge J \wedge \mathrm{Tr}(B^+_{2,0} \wedge F_{0,2}). \tag{73}$$

The symmetries of this action are

$$\begin{aligned}
&QA_m = \Psi_m \quad QA_{\bar{m}} = D_{\bar{m}}c\\
&QB^+_{mn} = -[c, B^+_{mn}], \quad QB_{\bar{m}\bar{n}+} = \left(D_{[\bar{m}}\Psi_{\bar{n}]}\right)^+ - [c, B^+_{\bar{m}\bar{n}}] + [{}^*B^{mn+}, \Phi]\\
&Q\Psi_m = 0, \quad Q\Psi_{\bar{m}} = D_{\bar{n}}\Phi - [c, \Psi_{\bar{n}}]\\
&Q\Phi = -[c, \Phi], \quad Qc = -\frac{1}{2}[c, c].
\end{aligned} \tag{74}$$

The quantization of this action can be worked out using the BV formalism, and is very similar to that already discussed in [4].

We have eight freedoms for gauge fixing the system $(B^+_{2,0}, B^+_{0,2}, A_{1,0}, A_{0,1})$. Indeed, Ψ_m, $\Psi_{\bar{m}}$ and c have respectively four, four and one components, but $\Psi_{\bar{m}}$ has a ghost of ghost symmetry with ghost of ghost Φ, so it only counts for $3 = 4 - 1$ freedoms. We can choose the following seven gauge-fixing conditions in the gauge covariant sector:

$$B^+_{mn} = \epsilon_{mnlp\bar{m}\bar{n}\bar{l}\bar{p}}F^{\bar{m}\bar{n}+}J^{l\bar{l}}J^{p\bar{p}} \tag{75}$$

[9] As discussed for the four-dimensional case, in order to recover the Yukawa couplings and the quartic term in the potential $[\bar{\Phi}, \Phi]^2$ typical of the $N = 2$ SYM one should slightly modify the gauge-fixing fermion (68).

$$
\begin{aligned}
&B_{\bar{m}\bar{n}^+} = 0 \\
&J^{m\bar{n}} F_{m\bar{n}} = 0
\end{aligned}
\tag{76}
$$

plus the ordinary transversality condition for the gauge field, $(\partial^m A_m + \partial^{\bar{m}} A_{\bar{m}} = 0)$. The transformation law of B^+_{mn} implies that a gauge fixing for $\Psi_{\bar{m}}$ must be also done, with a gauge function:

$$
D^{\bar{m}} \Psi_{\bar{m}} = 0. \tag{77}
$$

The Q-invariant gauge fixing of the action with these functions is standard, and reproduces the action as in [4], that is the twisted form of the eight-dimensional Super Yang–Mills theory. The classical action (73) can be considered as an eight-dimensional generalization of self-dual Yang–Mills in four dimensions, in particular of its realization studied in [22]. We should notice that we are at the extreme point of the definition of an ATQFT. The action is the sum of a d-closed term and a Q-exact term. Thus, there is a ring of observables defined from the cohomology of the BRST operator. However, the classical action is actually completely dependent on the metrics of the manifold, since it depends on both the Kähler form and the complex form at the same time.

6. BF theory and Topological M theory

Let us consider the following real BF model on a six-manifold M

$$
S_{BF,\alpha} = \int_M \Phi \wedge \mathrm{Tr}\left(B F_A + \frac{\alpha^2}{3} B^3 \right) \tag{78}
$$

where Φ is a real 3-form, B a 1-form and A a gauge connection with curvature F_A. Both A and B are valued in the adjoint representation of an unitary gauge group. On a Calabi–Yau manifold X, the partition function of this theory

$$
Z_{BF,\alpha}(X, \beta) = \int \mathcal{D}A\mathcal{D}B \mathrm{e}^{-\beta S_{BF,\alpha}} \tag{79}
$$

can be formally identified with the square of the partition function of holomorphic Chern–Simons

$$
Z_{hCS}(X, l) = \int \mathcal{D}\mathcal{A} \mathrm{e}^{-l S_{hCS}(\mathcal{A})} \tag{80}
$$

with

$$
S_{hCS}(\mathcal{A}) = \int \bar{\Omega} \wedge \mathrm{Tr}\left(\mathcal{A}\partial\mathcal{A} + \frac{2}{3}\mathcal{A}^3 \right) \tag{81}
$$

provided that we identify the real 3-form Φ as the imaginary part of the holomorphic (3, 0)-form Ω of the Calabi–Yau threefold and make the following identifications

$$
\begin{aligned}
A &= \frac{1}{2}(\mathcal{A} + \bar{\mathcal{A}}) \\
B &= -\frac{i}{2\alpha}(\mathcal{A} - \bar{\mathcal{A}})
\end{aligned}
\tag{82}
$$

and $\beta = 4l\alpha$. This is the generalization to six dimensions of the relationship already known in three dimensions [33] between BF theory (and the associated Turaev–Viro invariant [39]) and the Chern–Simons theory (and the associated Reshetikhin–Turaev invariant [40]).

The advantage of the action (78) is that on some special manifolds it can be directly related to gravitational theories. For example, one can lift (78) to a seven-dimensional manifold as

$$S_{BF,7} = \int_N G \wedge \mathrm{Tr}\left(BF_A + \frac{\alpha^2}{3}B^3\right). \tag{83}$$

On G_2 manifolds which can be described as the fibration of a spin bundle over a three-manifold (or a four-manifold), the term $(BF + B^3)$ can be identified with the associative 3-form of the Hitchin's action by considering the field B as the vielbein and the field A as the spin connection (see Section 6 of [12]). This allows us to relate the BF model (83) to Hitchin's action functional. Notice that if $N = X \times S^1$ we can take[10]

$$G = \mathrm{Im}(\Omega)\mathrm{d}t + \frac{1}{2}J \wedge J, \tag{84}$$

where t is the coordinate on S^1 and J the Kähler form on X. Then by choosing the temporal gauge $A_t = B_t = 0$ for the A and B fields one recovers the partition function (79), where now β is the size of the circle S^1.

Acknowledgments

L.B. is grateful to I.M. Singer for interesting discussions. A.T. would like to thank U. Bruzzo for useful discussions and all members of LPTHE where most of this work has been done.

References

[1] L. Baulieu, Gravitational topological quantum field theory versus $N = 2$ $D = 8$ supergravity and its lift to $N = 1$ $D = 11$ supergravity, JHEP 0404 (2004) 044. hep-th/0304221;
L. Baulieu, M. Bellon, A. Tanzini, Supergravity and the knitting of the Kalb–Ramond two-form in eight-dimensional topological gravity, Phys. Lett. B 565 (2003) 211. arXiv:hep-th/0303165; Eight-dimensional topological gravity and its correspondence with supergravity, Phys. Lett. B, 543 (2002) 291. arXiv:hep-th/0207020.

[2] D. Anselmi, P. Fre, Twisted $N = 2$ supergravity as topological gravity in four-dimensions, Nuclear Phys. B 392 (1993) 401. arXiv:hep-th/9208029;
L. Baulieu, A. Tanzini, Topological gravity versus supergravity on manifolds with special holonomy, JHEP 0203 (2002) 015. hep-th/0201109;
P. de Medeiros, B. Spence, Four-dimensional topological Einstein–Maxwell gravity, Classical Quantum Gravity 20 (2003) 2075. arXiv:hep-th/0209115.

[3] L. Baulieu, Going down from a 3-form in 16 dimensions, Phys. Lett. B 544 (2002) 367–373. hep-th/0207184.

[4] L. Baulieu, H. Kanno, I.M. Singer, Special quantum field theories in eight and other dimensions, Comm. Math. Phys. 194 (1998) 149. hep-th/9704167, hep-th/9705127;
L. Baulieu, C. Laroche, On generalized self-duality equations toward supersymmetric quantum field theories of forms, Modern Phys. Lett. A13 (1998) 1115. hep-th/9801014.

[5] B.S. Acharya, J.M. Figueroa-O'Farrill, B. Spence, M. O'Loughlin, Euclidean D-branes and higher-dimensional gauge theory, Nuclear Phys. B 514 (1998) 583. arXiv:hep-th/9707118; Higher-dimensional analogues of Donaldson–Witten theory, Nuclear Phys. B, 503 (1997) 657. arXiv:hep-th/9705138.

[6] M. Blau, G. Thompson, Euclidean SYM theories by time reduction and special holonomy manifolds, Phys. Lett. B 415 (1997) 242. arXiv:hep-th/9706225.

[7] A. Iqbal, N. Nekrasov, A. Okounkov, C. Vafa, Quantum foam and topological strings, hep-th/0312022.

[8] A. Neitzke, C. Vafa, $N = 2$ strings and the twistorial Calabi–Yau, hep-th/0402128.

[10] A similar analysis can be made if we replace the circle S^1 by the real line $\mathbb{R}$ or an interval. In the last case, boundary terms have to be taken into account.

[9] N. Nekrasov, H. Ooguri, C. Vafa, S-duality and topological strings, JHEP 0410 (2004) 009. hep-th/0403167.
[10] A. Kapustin, Gauge theory, topological strings, and S-duality, JHEP 0409 (2004) 034. hep-th/0404041.
[11] H. Ooguri, A. Strominger, C. Vafa, Black hole attractors and the topological string, hep-th/0405146.
[12] R. Dijkgraaf, S. Gukov, A. Neitzke, C. Vafa, Topological M-theory as unification of form theories of gravity, hep-th/0411073.
[13] A. Gerasimov, S.L. Shatashvili, Towards integrability of topological strings I: Three-forms on Calabi–Yau manifolds, hep-th/0409238.
[14] P.A. Grassi, P. Vanhove, Topological M theory from pure spinor formalism, hep-th/0411167.
[15] N. Hitchin, Stable forms and special metrics, in: M. Fernandez, J. Wolf (Eds.) Proceedings of the Congress in memory of Alfred Gray, in: AMS Contemporary Mathematics Series, math.DG/0107101; The geometry of three-forms in six and seven dimensions, math.DG/0010054.
[16] A.S. Cattaneo, P. Cotta-Ramusino, F. Fucito, M. Martellini, M. Rinaldi, A. Tanzini, M. Zeni, Four-dimensional Yang–Mills theory as a deformation of topological BF theory, Comm. Math. Phys. 197 (1998) 571. arXiv:hep-th/9705123.
[17] L. Baulieu, S.L. Shatashvili, Duality from topological symmetry, JHEP 9903 (1999) 011. arXiv:hep-th/9811198.
[18] D. Maulik, N. Nekrasov, A. Okounkov, R. Pandharipande, math.AG/0312059, math.AG/0406092.
[19] M.J. Plebanski, On the separation of Einsteinian substructures, J. Math. Phys. 18 (1977) 2511;
G.T. Horowitz, Exactly soluble diffeomorphism invariant theories, Comm. Math. Phys. 125 (1989) 417;
C.G. Torre, Perturbations of gravitational instantons, Phys. Rev. D 41 (1990) 3620;
M. Abe, A. Nakamichi, T. Ueno, Gravitational instantons and moduli spaces in topological two form gravity, Phys. Rev. D 50 (1994) 7323. hep-th/9408178.
[20] Work in progress.
[21] E. Witten, $(2+1)$-Dimensional gravity as an exactly soluble system, Nuclear Phys. B 311 (1988) 46.
[22] W. Siegel, $N=2$, $N=4$ string theory is selfdual $N=4$ Yang–Mills theory, Phys. Rev. D 46 (1992) 3235.
[23] L. Baulieu, I.M. Singer, Nuclear Phys. Proc. Suppl. B5 (1988) 12.
[24] A.D. Popov, Holomorphic analogs of topological gauge theories, Phys. Lett. B 473 (2000) 65. arXiv:hep-th/9909135;
T.A. Ivanova, A.D. Popov, Dressing symmetries of holomorphic BF theories, J. Math. Phys. 41 (2000) 2604. arXiv:hep-th/0002120.
[25] J.S. Park, $N=2$ topological Yang–Mills theory on compact Kahler surfaces, Comm. Math. Phys. 163 (1994) 113. arXiv:hep-th/9304060;
J.S. Park, Holomorphic Yang–Mills theory on compact Kahler manifolds, Nuclear Phys. B 423 (1994) 559. arXiv:hep-th/9305095.
[26] A. Johansen, Twisting of $N=1$ SUSY gauge theories and heterotic topological theories, Internat J. Modern Phys. A 10 (1995) 4325–4358. hep-th/9403017.
[27] E. Witten, Supersymmetric Yang–Mills Theory On A Four-Manifold, J. Math. Phys. 35 (1994) 5101–5135. hep-th/9403195.
[28] R. Dijkgraaf, J.S. Park, B.J. Schroers, $N=4$ supersymmetric Yang–Mills theory on a Kahler surface, arXiv:hep-th/9801066.
[29] C. Hofman, J.S. Park, Cohomological Yang–Mills theories on Kahler 3-folds, Nuclear Phys. B 600 (2001) 133. arXiv:hep-th/0010103.
[30] L. Baulieu, Algebraic Construction Of Gauge Invariant Theories, in: Particles and Fields, in: Cargese Summer Inst. 1983, vol. 1, Cargese; On forms with non-abelian charges and their dualities, Phys. Lett. B 441 (1998) 250. hep-th/9808055;
L. Baulieu, E. Rabinovici, Selfduality and new tqfts for forms, JHEP 9806 (1998) 0068.
[31] H.B. Lawson, M.-L. Michelsohn, Spin Geometry, Princeton University Press, Princeton, New Jersey, 1989.
[32] E. Witten, Chern–Simons gauge theory as a string theory, Prog. Math. 133 (1995) 637. arXiv:hep-th/9207094.
[33] A.S. Cattaneo, P. Cotta-Ramusino, J. Frohlich, M. Martellini, Topological BF theories in three-dimensions and four-dimensions, J. Math. Phys. 36 (1995) 6137. arXiv:hep-th/9505027.
[34] D. Ray, A. Singer, Analytic torsion for complex manifolds, Ann. Math. 98 (1973) 154–177.
[35] S.K. Donaldson, R.P. Thomas, Gauge theory in higher dimensions, Prepared for conference on Geometric Issues in Foundations of Science in honor of Sir Roger Penrose's 65th Birthday, Oxford, England, 25–29 June 1996;
R.P. Thomas, Ph.D. Thesis, Oxford, 1997.
[36] I. Frenkel, B. Khesin, A. Todorov, Complex counterpart of Chern–Simons–Witten theory and holomorphic linking (preprint).

[37] S. Axelrod, I.M. Singer, Chern–Simons perturbation theory, hep-th/9110056; J. Diff. Geom. 39 (1994) 173. arXiv:hep-th/9304087.
[38] P. de Bartolomeis, G. Tian, J. Diff. Geom. 43 (1996) 231.
[39] V.G. Turaev, O.Yu. Viro, Sum invariants of 3-manifolds and quantum 6j-symbols, Topology 31 (1992) 865.
[40] N. Reshetikhin, V.G. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups, Invent. Math. 103 (1991) 547.